

# Short-time asymptotics for marginal distributions of semimartingales

Amel Bentata and Rama Cont

January 2012

## Abstract

We study the short-time asymptotics of conditional expectations of smooth and non-smooth functions of a (discontinuous) Ito semimartingale; we compute the leading term in the asymptotics in terms of the local characteristics of the semimartingale. We derive in particular the asymptotic behavior of call options with short maturity in a semimartingale model: whereas the behavior of *out-of-the-money* options is found to be linear in time, the short time asymptotics of *at-the-money* options is shown to depend on the fine structure of the semimartingale.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Short time asymptotics for conditional expectations</b>	<b>4</b>
2.1	Main result . . . . .	4
2.2	Some consequences and examples . . . . .	9
2.2.1	Functions of a Markov process . . . . .	10
2.2.2	Time-changed Lévy processes . . . . .	12
<b>3</b>	<b>Short-maturity asymptotics for call options</b>	<b>13</b>
3.1	Out-of-the money call options . . . . .	14
3.2	At-the-money call options . . . . .	18

# 1 Introduction

In applications such as stochastic control, statistics of processes and mathematical finance, one is often interested in computing or approximating conditional expectations of the type

$$\mathbb{E}[f(\xi_t)|\mathcal{F}_{t_0}] \quad (1)$$

where  $\xi$  is a stochastic process. Whereas for Markov process various well-known tools –partial differential equations, Monte Carlo simulation, semigroup methods– are available for the computation and approximation of conditional expectations, such tools do not carry over to the more general setting of semimartingales. Even in the Markov case, if the state space is high dimensional exact computations may be computationally prohibitive and there has been a lot of interest in obtaining approximations of (1) as  $t \rightarrow t_0$ . Knowledge of such *short-time asymptotics* is very useful not only for computation of conditional expectations but also for the estimation and calibration of such models. Accordingly, short-time asymptotics for (1) (which, in the Markov case, amounts to studying transition densities of the process  $\xi$ ) has been previously studied for diffusion models [6, 8, 11], Lévy processes [15, 16, 21, 2, 13, 12, 22], Markov jump-diffusion models [1, 3] and one-dimensional martingales [17], using a variety of techniques. The proofs of these results in the case of Lévy processes makes heavy use of the independence of increments; proofs in other case rely on the Markov property, estimates for heat kernels for second-order differential operators or Malliavin calculus. What is striking, however, is the similarity of the results obtained in these different settings.

We reconsider here the short-time asymptotics of conditional expectations in a more general framework which contains existing models but allows to go beyond the Markovian setting and to incorporate path-dependent features. Such a framework is provided by the class of *Itô semimartingales*, which contains all the examples cited above but allows the use the tools of stochastic analysis. An *Itô semimartingale* on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  is a stochastic process  $\xi$  with the representation

$$\xi_t = \xi_0 + \int_0^t \beta_s ds + \int_0^t \delta_s dW_s + \int_0^t \int_{\mathbb{R}^d} \kappa(y) \tilde{M}(dsdy) + \int_0^t \int_{\mathbb{R}^d} (y - \kappa(y)) M(dsdy), \quad (2)$$

where  $\xi_0$  is in  $\mathbb{R}^d$ ,  $W$  is a standard  $\mathbb{R}^n$ -valued Wiener process,  $M$  is an integer-valued random measure on  $[0, \infty] \times \mathbb{R}^d$  with compensator  $\mu(\omega, dt, dy) = m(\omega, t, dy)dt$  and  $\tilde{M} = M - \mu$  its compensated random measure,  $\beta$  (resp.  $\delta$ ) is an adapted process with values in  $\mathbb{R}^d$  (resp.  $M_{d \times n}(\mathbb{R})$ ) and

$$\kappa(y) = \frac{y}{1 + \|y\|^2}$$

is a truncation function.

We study the short-time asymptotics of conditional expectations of the form (1) where  $\xi$  is an Itô semimartingale of the form (2), for various classes of

functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . First, we prove a general result for the case of  $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$ . Then we will treat, when  $d = 1$ , the case of

$$\mathbb{E}[(\xi_t - K)^+ | \mathcal{F}_{t_0}], \quad (3)$$

which corresponds to the value at  $t_0$  of a call option with strike  $K$  and maturity  $t$  in a model described by equation (2). We show that whereas the behavior of (3) in the case  $K > \xi_{t_0}$  (*out-of-the-money* options) is linear in  $t - t_0$ , the asymptotics in the case  $K = \xi_{t_0}$  (which corresponds to *at-the-money* options) depends on the fine structure of the semimartingale  $\xi$  at  $t_0$ . In particular, we show that for continuous semimartingales the short-maturity asymptotics of at-the-money options is determined by the local time of  $\xi$  at  $t_0$ . In each case we identify the leading term in the asymptotics and express this term in terms of the local characteristics of the semimartingale at  $t_0$ .

Our results unify various asymptotic results previously derived for particular examples of stochastic models and extend them to the more general case of a discontinuous semimartingale. In particular, we show that the independence of increments or the Markov property do not play any role in the derivation of such results.

Short-time asymptotics for expectations of the form (1) have been studied in the context of statistics of processes [15, 2] and option pricing [1, 6, 8, 3, 14, 13, 12, 22, 17]. Berestycki, Busca and Florent [6, 8] and Gatheral et al [14] derive short maturity asymptotics for call options when  $\xi_t$  is a diffusion, using analytical methods. Durrleman [10] studied the asymptotics of implied volatility in a stochastic volatility model. Jacod [15] derived asymptotics for (1) for various classes of functions  $f$ , when  $\xi_t$  is a Lévy process. Figueroa-Lopez and Forde [13] and Tankov [22] study the asymptotics of (3) when  $\xi_t$  is the exponential of a Lévy process. Figueroa-Lopez and Houdré [13] also studies short-time asymptotic expansions for (1), by iterating the infinitesimal generator of the Lévy process  $\xi_t$ . Figueroa-Lopez and Forde [13] extend these results and derive a second order small-time expansion for out-of-the-money call options under an exponential Lévy model. Alos et al [1] derive short-maturity expansions for call options and implied volatility in a Heston model using Malliavin calculus. Benhamou et al. [3] derive short-maturity expansions for call options in a model where  $\xi$  is a Markov process whose jumps are described by a compound Poisson process. More generally, these results apply to processes with independent increments or Markov processes expressed as the solution of a stochastic differential equation with regular coefficients.

Durrleman studied the convergence of implied volatility to spot volatility in a stochastic volatility model with finite-variation jumps [9]. More recently, Nutz and Muhle-Karbe [17] study short-maturity asymptotics for call options in the case where  $\xi_t$  is a one-dimensional Itô semimartingale driven by a (one-dimensional) Poisson random measure whose Lévy measure is absolutely continuous. Their approach consists in “freezing” the characteristic triplet of  $\xi$  at  $t_0$ , approximating  $\xi_t$  by the corresponding Lévy process and using the results cited above [15, 13] to derive asymptotics for call option prices.

Our contribution is to extend these results to the more general case when  $\xi$  is a  $d$ -dimensional semimartingale with jumps. By using minimal assumptions on the process  $\xi$ , we put previous results into perspective: in contrast to previous derivations, our approach is purely based on Itô calculus and makes no use of the Markov property or independence of increments. Also, our multidimensional setting allows to treat examples which are not accessible using previous results such as [17]. For instance, when studying index options in jump-diffusion models, one considers an index  $I_t = \sum w_i S_t^i$  where  $(S^1, \dots, S^d)$  are Itô semimartingales. In this framework,  $I$  is indeed an Itô semimartingale whose stochastic integral representation is implied by those of  $S^i$  but it is naturally represented in terms of a  $d$ -dimensional integer-valued random measure, not a one-dimensional Poisson random measure. Our setting provides a natural framework for treating such examples.

## 2 Short time asymptotics for conditional expectations

### 2.1 Main result

We make the following assumptions on the characteristics of the semimartingale  $\xi$ :

**Assumption 2.1** (Right-continuity of characteristics at  $t_0$ ).

$$\lim_{t \rightarrow t_0, t > t_0} \mathbb{E} [\|\beta_t - \beta_{t_0}\| | \mathcal{F}_{t_0}] = 0, \quad \lim_{t \rightarrow t_0, t > t_0} \mathbb{E} [\|\delta_t - \delta_{t_0}\|^2 | \mathcal{F}_{t_0}] = 0,$$

where  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^d$  and for  $\varphi \in \mathcal{C}_0^b(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ ,

$$\lim_{t \rightarrow t_0, t > t_0} \mathbb{E} \left[ \int_{\mathbb{R}^d} \|y\|^2 \varphi(\xi_t, y) m(t, dy) | \mathcal{F}_{t_0} \right] = \int_{\mathbb{R}^d} \|y\|^2 \varphi(\xi_{t_0}, y) m(t_0, dy).$$

The second requirement, which may be viewed as a weak (right) continuity of  $m(t, dy)$  along the paths of  $\xi$ , is satisfied for instance if  $m(t, dy)$  is absolutely continuous with a density which is right-continuous in  $t$  at  $t_0$ .

**Assumption 2.2** (Integrability condition).  $\exists T > t_0$ ,

$$\begin{aligned} \mathbb{E} \left[ \int_{t_0}^T \|\beta_s\| ds | \mathcal{F}_{t_0} \right] &< \infty, & \mathbb{E} \left[ \int_{t_0}^T \|\delta_s\|^2 ds | \mathcal{F}_{t_0} \right] &< \infty, \\ \mathbb{E} \left[ \int_{t_0}^T \int_{\mathbb{R}^d} \|y\|^2 m(s, dy) ds | \mathcal{F}_{t_0} \right] &< \infty. \end{aligned}$$

Under these assumptions, the following result describes the asymptotic behavior of  $\mathbb{E}[f(\xi_t) | \mathcal{F}_{t_0}]$  when  $t \rightarrow t_0$ :

**Theorem 2.1.** Under Assumptions 2.1 and 2.2, for all  $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$ ,

$$\lim_{t \downarrow t_0} \frac{1}{t - t_0} (\mathbb{E}[f(\xi_t) | \mathcal{F}_{t_0}] - f(\xi_{t_0})) = \mathcal{L}_{t_0} f(\xi_{t_0}). \quad (4)$$

where  $\mathcal{L}_{t_0}$  is the (random) integro-differential operator given by

$$\begin{aligned} \forall f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{L}_{t_0} f(x) &= \beta_{t_0} \cdot \nabla f(x) + \frac{1}{2} \text{tr} \left[ {}^t \delta_{t_0} \delta_{t_0} \nabla^2 f \right] (x) \\ &+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1 + \|y\|^2} y \cdot \nabla f(x)] m(t_0, dy). \end{aligned} \quad (5)$$

Before proving Theorem 2.1, we recall a useful lemma:

**Lemma 2.1.** Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  be right-continuous at 0, then

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t f(s) ds = f(0). \quad (6)$$

*Proof.* Let  $F$  denote the primitive of  $f$ , then

$$\frac{1}{t} \int_0^t f(s) ds = \frac{1}{t} (F(t) - F(0)).$$

Letting  $t \rightarrow 0^+$ , this is nothing but the right derivative at 0 of  $F$ , which is  $f(0)$  by right continuity of  $f$ .  $\square$

We can now prove Theorem 2.1.

*Proof.* of Theorem 2.1

We first note that, by replacing  $\mathbb{P}$  by the conditional measure  $\mathbb{P}_{|\mathcal{F}_{t_0}}$  given  $\mathcal{F}_{t_0}$ , we may replace the conditional expectation in (4) by an expectation with respect to the marginal distribution of  $\xi_t$  under  $\mathbb{P}_{|\mathcal{F}_{t_0}}$ . Thus, without loss of generality, we put  $t_0 = 0$  in the sequel and consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets. Let  $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$ . Itô's formula yields

$$\begin{aligned} f(\xi_t) &= f(\xi_0) + \int_0^t \nabla f(\xi_{s-}) d\xi_s^i + \frac{1}{2} \int_0^t \text{tr} [\nabla^2 f(\xi_{s-}) {}^t \delta_s \delta_s] ds \\ &+ \sum_{s \leq t} \left[ f(\xi_{s-} + \Delta \xi_s) - f(\xi_{s-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\xi_{s-}) \Delta \xi_s^i \right] \\ &= f(\xi_0) + \int_0^t \nabla f(\xi_{s-}) \cdot \beta_s ds + \int_0^t \nabla f(\xi_{s-}) \cdot \delta_s dW_s \\ &+ \frac{1}{2} \int_0^t \text{tr} [\nabla^2 f(\xi_{s-}) {}^t \delta_s \delta_s] ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \nabla f(\xi_{s-}) \cdot \kappa(y) \tilde{M}(ds dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} (f(\xi_{s-} + y) - f(\xi_{s-}) - \kappa(y) \cdot \nabla f(\xi_{s-})) M(ds dy). \end{aligned}$$

We note that

- since  $\nabla f$  is bounded and given Assumption 2.2,  $\int_0^t \int_{\mathbb{R}^d} \nabla f(\xi_{s-}) \cdot \kappa(y) \tilde{M}(ds dy)$  is a square-integrable martingale.
- since  $\nabla f$  is bounded and given Assumption 2.2,  $\int_0^t \nabla f(\xi_{s-}) \cdot \delta_s dW_s$  is a martingale.

Hence, taking expectations, we obtain

$$\begin{aligned}
\mathbb{E}[f(\xi_t)] &= \mathbb{E}[f(\xi_0)] + \mathbb{E}\left[\int_0^t \nabla f(\xi_{s-}) \cdot \beta_s ds\right] + \mathbb{E}\left[\frac{1}{2} \int_0^t \text{tr}[\nabla^2 f(\xi_{s-})^t \delta_s \delta_s] ds\right] \\
&+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} (f(\xi_{s-} + y) - f(\xi_{s-}) - \kappa(y) \cdot \nabla f(\xi_{s-})) M(ds dy)\right] \\
&= \mathbb{E}[f(\xi_0)] + \mathbb{E}\left[\int_0^t \nabla f(\xi_{s-}) \cdot \beta_s ds\right] + \mathbb{E}\left[\frac{1}{2} \int_0^t \text{tr}[\nabla^2 f(\xi_{s-})^t \delta_s \delta_s] ds\right] \\
&+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} (f(\xi_{s-} + y) - f(\xi_{s-}) - \kappa(y) \cdot \nabla f(\xi_{s-})) m(s, dy) ds\right],
\end{aligned}$$

that is

$$\mathbb{E}[f(\xi_t)] = \mathbb{E}[f(\xi_0)] + \mathbb{E}\left[\int_0^t \mathcal{L}_s f(\xi_s) ds\right]. \quad (7)$$

where  $\mathcal{L}$  denote the integro-differential operator given, for all  $t \in [t_0, T]$  and for all  $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$ , by

$$\begin{aligned}
\mathcal{L}_t f(x) &= \beta_t \cdot \nabla f(x) + \frac{1}{2} \text{tr}[\delta_t \delta_t \nabla^2 f](x) \\
&+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1+\|y\|^2} y \cdot \nabla f(x)] m(t, dy),
\end{aligned} \quad (8)$$

Equation (7) yields

$$\begin{aligned}
&\frac{1}{t} \mathbb{E}[f(\xi_t)] - \frac{1}{t} f(\xi_0) - \mathcal{L}_0 f(\xi_0) \\
&= \mathbb{E}\left[\frac{1}{t} \int_0^t ds (\nabla f(\xi_s) \cdot \beta_s - \nabla f(\xi_0) \cdot \beta_0)\right] \\
&+ \frac{1}{2} \mathbb{E}\left[\frac{1}{t} \int_0^t ds \text{tr}[\nabla^2 f(\xi_s)^t \delta_s \delta_s - \nabla^2 f(\xi_0)^t \delta_0 \delta_0]\right] \\
&+ \mathbb{E}\left[\int_{\mathbb{R}^d} \frac{1}{t} \int_0^t ds [m(s, dy) (f(\xi_s + y) - f(\xi_s) - \kappa(y) \cdot \nabla f(\xi_s)) \right. \\
&\quad \left. - m(0, dy) (f(\xi_0 + y) - f(\xi_0) - \kappa(y) \cdot \nabla f(\xi_0))]\right].
\end{aligned}$$

Define

$$\begin{aligned}
\Delta_1(t) &= \mathbb{E} \left[ \frac{1}{t} \int_0^t ds (\nabla f(\xi_s) \cdot \beta_s - \nabla f(\xi_0) \cdot \beta_0) \right], \\
\Delta_2(t) &= \frac{1}{2} \mathbb{E} \left[ \frac{1}{t} \int_0^t ds \operatorname{tr} [\nabla^2 f(\xi_{s-})^t \delta_s \delta_s - \nabla^2 f(\xi_0)^t \delta_0 \delta_0] \right], \\
\Delta_3(t) &= \mathbb{E} \left[ \int_{\mathbb{R}^d} \frac{1}{t} \int_0^t ds [m(s, dy) (f(\xi_s + y) - f(\xi_s) - \kappa(y) \cdot \nabla f(\xi_{s-})) \right. \\
&\quad \left. - m(0, dy) (f(\xi_0 + y) - f(\xi_0) - \kappa(y) \cdot \nabla f(\xi_0)) \right].
\end{aligned}$$

Thanks to Assumptions 2.1 and 2.2,

$$\mathbb{E} \left[ \int_0^t ds |\nabla f(\xi_s) \cdot \beta_s - \nabla f(\xi_0) \cdot \beta_0| \right] \leq \mathbb{E} \left[ \int_0^t ds \|\nabla f\| (\|\beta_s\| + \|\beta_0\|) \right] < \infty.$$

Fubini's theorem then applies:

$$\Delta_1(t) = \frac{1}{t} \int_0^t ds \mathbb{E} [\nabla f(\xi_s) \cdot \beta_s - \nabla f(\xi_0) \cdot \beta_0].$$

Let us prove that

$$\begin{aligned}
g_1 &: [0, T[ \rightarrow \mathbb{R} \\
t &\rightarrow \mathbb{E} [\nabla f(\xi_t) \cdot \beta_t - \nabla f(\xi_0) \cdot \beta_0],
\end{aligned}$$

is right-continuous at 0 with  $g_1(0) = 0$ , yielding  $\Delta_1(t) \rightarrow 0$  when  $t \rightarrow 0^+$  if one applies Lemma 2.1.

$$\begin{aligned}
|g_1(t)| &= |\mathbb{E} [\nabla f(\xi_t) \cdot \beta_t - \nabla f(\xi_0) \cdot \beta_0]| \\
&= |\mathbb{E} [(\nabla f(\xi_t) - \nabla f(\xi_0)) \cdot \beta_0 + \nabla f(\xi_t) \cdot (\beta_t - \beta_0)]| \\
&\leq \|\nabla f\|_\infty \mathbb{E} [\|\beta_t - \beta_0\|] + \|\beta_0\| \|\nabla^2 f\|_\infty \mathbb{E} [\|\xi_t - \xi_0\|],
\end{aligned} \tag{9}$$

where  $\|\cdot\|_\infty$  denotes the supremum norm on  $\mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$ . Assumption 2.1 implies that:

$$\lim_{t \rightarrow 0^+} \mathbb{E} [\|\beta_t - \beta_0\|] = 0.$$

Thanks to Assumption 2.2, one may decompose  $\xi_t$  as follows

$$\begin{aligned}
\xi_t &= \xi_0 + A_t + M_t, \\
A_t &= \int_0^t \left( \beta_s ds + \int_{\mathbb{R}^d} (y - \kappa(y)) m(s, dy) \right) ds, \\
M_t &= \int_0^t \delta_s dW_s + \int_0^t \int_{\mathbb{R}^d} y \tilde{M}(ds dy),
\end{aligned} \tag{10}$$

where  $A_t$  is of finite variation and  $M_t$  is a local martingale. First, applying Fubini's theorem (using Assumption 2.2),

$$\begin{aligned}\mathbb{E} [\|A_t\|] &\leq \mathbb{E} \left[ \int_0^t \|\beta_s\| ds \right] + \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \|y - \kappa(y)\| m(s, dy) ds \right] \\ &= \int_0^t ds \mathbb{E} [\|\beta_s\|] + \int_0^t ds \mathbb{E} \left[ \int_{\mathbb{R}^d} \|y - \kappa(y)\| m(s, dy) \right].\end{aligned}$$

Thanks to Assumption 2.1, one observes that if  $s \in [0, T[ \rightarrow \mathbb{E} [\|\beta_s - \beta_0\|]$  is right-continuous at 0 so is  $s \in [0, T[ \rightarrow \mathbb{E} [\|\beta_s\|]$ . Furthermore, Assumption 2.1 yields that

$$s \in [0, T[ \rightarrow \mathbb{E} \left[ \int_{\mathbb{R}^d} \|y - \kappa(y)\| m(s, dy) \right]$$

is right-continuous at 0 and Lemma 2.1 implies that

$$\lim_{t \rightarrow 0^+} \mathbb{E} [\|A_t\|] = 0.$$

Furthermore, writing  $M_t = (M_t^1, \dots, M_t^d)$ ,

$$\mathbb{E} [\|M_t\|^2] = \sum_{1 \leq i \leq d} \mathbb{E} [M_t^i]^2.$$

Burkholder's inequality [19, Theorem IV.73] implies that there exists  $C > 0$  such that

$$\begin{aligned}\sup_{s \in [0, t]} \mathbb{E} [M_s^i]^2 &\leq C \mathbb{E} [M^i, M^i]_t \\ &= C \mathbb{E} \left[ \int_0^t ds |\delta_s^i|^2 + \int_0^t ds \int_{\mathbb{R}^d} |y_i|^2 m(s, dy) \right].\end{aligned}$$

Using Assumption 2.2 we may apply Fubini's theorem to obtain

$$\begin{aligned}\sup_{s \in [0, t]} \mathbb{E} [\|M_t\|^2] &\leq C \sum_{1 \leq i \leq d} \mathbb{E} \left[ \int_0^t ds |\delta_s^i|^2 \right] + \mathbb{E} \left[ \int_0^t ds \int_{\mathbb{R}^d} |y_i|^2 m(s, dy) \right] \\ &= C \left( \mathbb{E} \left[ \int_0^t ds \|\delta_s\|^2 \right] + \mathbb{E} \left[ \int_0^t ds \int_{\mathbb{R}^d} \|y\|^2 m(s, dy) \right] \right) \\ &= C \left( \int_0^t ds \mathbb{E} [\|\delta_s\|^2] + \int_0^t ds \mathbb{E} \left[ \int_{\mathbb{R}^d} \|y\|^2 m(s, dy) \right] \right).\end{aligned}$$

Thanks to Assumption 2.1, Lemma 2.1 yields

$$\lim_{t \rightarrow 0^+} \mathbb{E} [\|M_t\|^2] = 0.$$

Using the Jensen inequality, one obtains

$$\mathbb{E} [\|M_t\|] = \mathbb{E} \left[ \sqrt{\sum_{1 \leq i \leq d} M_t^i^2} \right] \leq \sqrt{\mathbb{E} \left[ \sum_{1 \leq i \leq d} M_t^i^2 \right]} = \mathbb{E} [\|M_t\|^2].$$



Hence,

$$\lim_{t \rightarrow 0^+} \mathbb{E} [\|M_t\|] = 0,$$

and

$$\lim_{t \rightarrow 0^+} \mathbb{E} [\|\xi_t - \xi_0\|] \leq \lim_{t \rightarrow 0^+} \mathbb{E} [\|A_t\|] + \lim_{t \rightarrow 0^+} \mathbb{E} [\|M_t\|] = 0.$$

Going back to the inequalities (9), one obtains

$$\lim_{t \rightarrow 0^+} g_1(t) = 0.$$

Similarly,  $\Delta_2(t) \rightarrow 0$  and  $\Delta_3(t) \rightarrow 0$  when  $t \rightarrow 0^+$ . This ends the proof.  $\square$

**Remark 2.1.** *In applications where a process is constructed as the solution to a stochastic differential equation driven by a Brownian motion and a Poisson random measure, one usually starts from a representation of the form*

$$\zeta_t = \zeta_0 + \int_0^t \beta_s ds + \int_0^t \delta_s dW_s + \int_0^t \int \psi_s(y) \tilde{N}(ds dy), \quad (11)$$

where  $\xi_0 \in \mathbb{R}^d$ ,  $W$  is a standard  $\mathbb{R}^n$ -valued Wiener process,  $\beta$  and  $\delta$  are non-anticipative càdlàg processes,  $N$  is a Poisson random measure on  $[0, T] \times \mathbb{R}^d$  with intensity  $\nu(dy) dt$  where  $\nu$  is a Lévy measure

$$\int_{\mathbb{R}^d} (1 \wedge \|y\|^2) \nu(dy) < \infty, \quad \tilde{N} = N - \nu(dy)dt,$$

and  $\psi : [0, T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$  is a predictable random function representing jump amplitude. This representation is different from (2), but [5, Lemma 2] shows that one can switch from the representation (11) to the representation (2) in an explicit manner.

In particular, if one rewrites Assumption 2.1 in the framework of equation (11), one recovers the Assumptions of [17] as a special case.

**Remark 2.2.** *It is sufficient for  $f$  to be locally bounded on the neighborhood of  $\xi_0$ .*

## 2.2 Some consequences and examples

If we have further information on the behavior of  $f$  in the neighborhood of  $\xi_0$ , then the quantity  $L_0 f(\xi_0)$  can be computed more explicitly. We summarize some commonly encountered situations in the following Proposition.

**Proposition 2.1.** *Under Assumptions 2.1 and 2.2,*

1. *If  $f(\xi_0) = 0$  and  $\nabla f(\xi_0) = 0$ , then*

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} [f(\xi_t)] = \frac{1}{2} \text{tr} [\delta_0 \delta_0 \nabla^2 f(\xi_0)] + \int_{\mathbb{R}^d} f(\xi_0 + y) m(0, dy). \quad (12)$$

2. If furthermore  $\nabla^2 f(\xi_0) = 0$ , then

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} [f(\xi_t)] = \int_{\mathbb{R}^d} f(\xi_0 + y) m(0, dy). \quad (13)$$

*Proof.* Applying Theorem 2.1,  $\mathcal{L}_0 f(\xi_0)$  writes

$$\begin{aligned} \mathcal{L}_0 f(\xi_0) &= \beta_0 \cdot \nabla f(\xi_0) + \frac{1}{2} \text{tr} [\nabla^2 f(\xi_0) {}^t \delta_0 \delta_0] (\xi_0) \\ &+ \int_{\mathbb{R}^d} [f(\xi_0 + y) - f(\xi_0) - \frac{1}{1 + \|y\|^2} y \cdot \nabla f(\xi_0)] m(0, dy). \end{aligned}$$

The proposition follows immediately.  $\square$

**Remark 2.3.** As observed by Jacod [15, Section 5.8] in the setting of Lévy processes, if  $f(\xi_0) = 0$  and  $\nabla f(\xi_0) = 0$ , then  $f(x) = O(\|x - \xi_0\|^2)$ . If furthermore  $\nabla^2 f(\xi_0) = 0$ , then  $f(x) = o(\|x - \xi_0\|^2)$ .

Let us now compute in a more explicit manner the asymptotics of (1) for specific semimartingales.

### 2.2.1 Functions of a Markov process

An important situations which often arises in applications is when a stochastic processe  $\xi$  is driven by an underlying Markov process, i.e.

$$\xi_t = f(Z_t) \quad f \in C^2(\mathbb{R}^d, \mathbb{R}), \quad (14)$$

where  $Z_t$  is a Markov process, defined as the weak solution on  $[0, T]$  of a stochastic differential equation

$$\begin{aligned} Z_t &= Z_0 + \int_0^t b(u, Z_{u-}) du + \int_0^t \Sigma(u, Z_{u-}) dW_u \\ &+ \int_0^t \int \psi(u, Z_{u-}, y) \tilde{N}(du dy), \end{aligned} \quad (15)$$

where  $(W_t)$  is an  $n$ -dimensional Brownian motion,  $N$  is a Poisson random measure on  $[0, T] \times \mathbb{R}^d$  with Lévy measure  $\nu(y) dy$ ,  $\tilde{N}$  the associated compensated random measure,  $\Sigma : [0, T] \times \mathbb{R}^d \mapsto M_{d \times d}(\mathbb{R})$ ,  $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$  and  $\psi : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$  are measurable functions such that

$$\begin{aligned} \psi(., ., 0) &= 0 \quad \psi(t, z, .) \text{ is a } \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d) - \text{diffeomorphism} \\ \forall t \in [0, T], \quad \mathbb{E} \left[ \int_0^t \int_{\{\|y\| \geq 1\}} \sup_{z \in \mathbb{R}^d} (1 \wedge \|\psi(s, z, y)\|^2) \nu(y) dy ds \right] &< \infty. \end{aligned} \quad (16)$$

In this setting, as shown in [5], one may verify the regularity assumptions Assumption 2.1 and Assumption 2.2 by requiring mild and easy-to-check assumptions on the coefficients:

**Assumption 2.3.**  $b(.,.), \Sigma(.,.)$  and  $\psi(.,.,y)$  are continuous in the neighborhood of  $(0, Z_0)$

**Assumption 2.4.** There exist  $T > 0, R > 0$  such that

$$\begin{aligned} \text{Either} \quad & \forall t \in [0, T] \quad \inf_{\|z - Z_0\| \leq R} \inf_{x \in \mathbb{R}^d, \|x\|=1} {}^t x \cdot \Sigma(t, z) \cdot x > 0 \\ \text{or} \quad & \Sigma \equiv 0. \end{aligned}$$

We then obtain the following result:

**Proposition 2.2.** Let  $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$  such that

$$\forall z \in \mathbb{R}^d, \quad \frac{\partial f}{\partial z_d}(z) \neq 0. \quad (17)$$

Define

$$\begin{cases} \beta_0 &= \nabla f(Z_0) \cdot b(0, Z_0) + \frac{1}{2} \text{tr} [\nabla^2 f(Z_0) {}^t \Sigma(0, Z_0) \Sigma(0, Z_0)] \\ &+ \int_{\mathbb{R}^d} (f(Z_0 + \psi(0, Z_0, y)) - f(Z_0) - \psi(0, Z_0, y) \cdot \nabla f(Z_0)) \nu(y) dy, \\ \delta_0 &= \|\nabla f(Z_0) \Sigma(0, Z_0)\|, \end{cases}$$

and the measure  $m(0, .)$  via

$$\begin{aligned} m(0, [u, \infty]) &= \int_{\mathbb{R}^d} 1_{\{f(Z_0 + \psi(0, Z_0, y)) - f(Z_0) \geq u\}} \nu(y) dy \quad u > 0, \\ m(0, [-\infty, u]) &= \int_{\mathbb{R}^d} 1_{\{f(Z_0 + \psi(0, Z_0, y)) - f(Z_0) \leq u\}} \nu(y) dy \quad u < 0. \end{aligned} \quad (18)$$

Under the Assumptions 2.3 and 2.4,  $\forall g \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$ ,

$$\lim_{t \rightarrow 0^+} \frac{\mathbb{E}[g(\xi_t)] - g(\xi_0)}{t} = \beta_0 g'(\xi_0) + \frac{\delta_0^2}{2} g''(\xi_0) + \int_{\mathbb{R}^d} [g(\xi_0 + u) - g(\xi_0) - u g'(\xi_0)] m(0, du). \quad (19)$$

*Proof.* Under the conditions (16) and the Assumption 2.4, Proposition ?? shows that  $\xi_t$  admits the semimartingale decomposition

$$\xi_t = \xi_0 + \int_0^t \beta_s ds + \int_0^t \delta_s dB_s + \int_0^t \int u \tilde{K}(ds du),$$

where

$$\begin{cases} \beta_t &= \nabla f(Z_{t-}) \cdot b(t, Z_{t-}) + \frac{1}{2} \text{tr} [\nabla^2 f(Z_{t-}) {}^t \Sigma(t, Z_{t-}) \Sigma(t, Z_{t-})] \\ &+ \int_{\mathbb{R}^d} (f(Z_{t-} + \psi(t, Z_{t-}, y)) - f(Z_{t-}) - \psi(t, Z_{t-}, y) \cdot \nabla f(Z_{t-})) \nu(y) dy, \\ \delta_t &= \|\nabla f(Z_{t-}) \Sigma(t, Z_{t-})\|, \end{cases}$$

and  $K$  is an integer-valued random measure on  $[0, T] \times \mathbb{R}$  with compensator  $k(t, Z_{t-}, u) du dt$  defined via

$$k(t, Z_{t-}, u) = \int_{\mathbb{R}^{d-1}} |\det \nabla_y \Phi(t, Z_{t-}, (y_1, \dots, y_{d-1}, u))| \nu(\Phi(t, Z_{t-}, (y_1, \dots, y_{d-1}, u))) dy_1 \cdots dy_{d-1},$$

with

$$\begin{cases} \Phi(t, z, y) = \phi(t, z, \kappa_z^{-1}(y)) & \kappa_z^{-1}(y) = (y_1, \dots, y_{d-1}, F_z(y)), \\ F_z(y) : \mathbb{R}^d \rightarrow \mathbb{R} & f(z + (y_1, \dots, y_{d-1}, F_z(y))) - f(z) = y_d. \end{cases}$$

From Assumption 2.3 it follows that Assumptions 2.1 and 2.2 hold for  $\beta_t, \delta_t$  and  $k(t, Z_{t-}, \cdot)$  on  $[0, T]$ . Applying Theorem 2.1, the result follows immediately.  $\square$

**Remark 2.4.** Benhamou et al. [3] studied the case where  $Z_t$  is the solution of a ‘Markovian’ SDE whose jumps are given by a compound Poisson Process. The above results generalizes their result to the (general) case where the jumps are driven by an arbitrary integer-valued random measure.

### 2.2.2 Time-changed Lévy processes

Models based on time-changed Lévy processes provide another class of examples of non-Markovian models which have generated recent interest in mathematical finance. Let  $L_t$  be a real-valued Lévy process,  $(b, \sigma^2, \nu)$  be its characteristic triplet,  $N$  its jump measure. Define

$$\xi_t = L_{\Theta_t} \quad \Theta_t = \int_0^t \theta_s ds, \quad (20)$$

where  $(\theta_t)$  is a locally bounded  $\mathcal{F}_t$ -adapted positive càdlàg process, interpreted as the rate of time change.

**Proposition 2.3.** *If*

$$\int_{\mathbb{R}} |y|^2 \nu(dy) < \infty \quad \text{and} \quad \lim_{t \rightarrow 0, t > 0} \mathbb{E} [|\theta_t - \theta_0|] = 0 \quad (21)$$

then

$$\forall f \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R}), \quad \lim_{t \rightarrow 0^+} \frac{\mathbb{E} [f(\xi_t)] - f(\xi_0)}{t} = \theta_0 \mathcal{L}_0 f(\xi_0) \quad (22)$$

where  $\mathcal{L}_0$  is the infinitesimal generator of the  $L$ :

$$\mathcal{L}_0 f(x) = b f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1+|y|^2} y f'(x)] \nu(dy). \quad (23)$$

*Proof.* Considering the Lévy-Itô decomposition of  $L$ :

$$\begin{aligned} L_t &= \left( b - \int_{\{|y| \leq 1\}} (y - \kappa(y)) \nu(dy) \right) t + \sigma W_t \\ &+ \int_0^t \int_{\mathbb{R}} \kappa(z) \tilde{N}(ds dz) + \int_0^t \int_{\mathbb{R}} (z - \kappa(z)) N(ds dz), \end{aligned}$$

then, as shown in [5],  $\xi$  has the representation

$$\begin{aligned} \xi_t &= \xi_0 + \int_0^t \sigma \sqrt{\theta_s} dZ_s + \int_0^t \left( b - \int_{\{|y| \leq 1\}} (y - \kappa(y)) \nu(dy) \right) \theta_s ds \\ &+ \int_0^t \int_{\mathbb{R}} \kappa(z) \theta_s \tilde{N}(ds dz) + \int_0^t \int_{\mathbb{R}} (z - \kappa(z)) \theta_s N(ds dz). \end{aligned}$$

where  $Z$  is a Brownian motion. With the notation of equation (2), one identifies

$$\beta_t = \left( b - \int_{\{|y| \leq 1\}} (y - \kappa(y)) \nu(dy) \right) \theta_t, \quad \delta_t = \sigma \sqrt{\theta_t}, \quad m(t, dy) = \theta_t \nu(dy).$$

If (21) holds, then Assumptions 2.1 and 2.2 hold for  $(\beta, \delta, m)$  and Theorem 2.1 may be applied to obtain the result.  $\square$

### 3 Short-maturity asymptotics for call options

Consider a (strictly positive) price process  $S$  whose dynamics under the pricing measure  $\mathbb{P}$  is given by a stochastic volatility model with jumps:

$$S_t = S_0 + \int_0^t r(s) S_{s-} ds + \int_0^t S_{s-} \delta_s dW_s + \int_0^t \int_{-\infty}^{+\infty} S_{s-} (e^y - 1) \tilde{M}(ds dy), \quad (24)$$

where  $r(t) > 0$  represents a (deterministic) bounded discount rate. For convenience, we shall assume that  $r \in \mathcal{C}_0^b(\mathbb{R}^+, \mathbb{R}^+)$ .  $\delta_t$  represents the volatility process and  $M$  is an integer-valued random measure with compensator  $\mu(\omega; dt dy) = m(\omega; t, dy) dt$ , representing jumps in the log-price, and  $\tilde{M} = M - \mu$  its compensated random measure. We make the following assumptions on the characteristics of  $S$ :

**Assumption 3.1** (Right-continuity at  $t_0$ ).

$$\lim_{t \rightarrow t_0, t > t_0} \mathbb{E} [|\delta_t - \delta_{t_0}|^2 | \mathcal{F}_{t_0}] = 0.$$

For all  $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ ,

$$\lim_{t \rightarrow t_0, t > t_0} \mathbb{E} \left[ \int_{\mathbb{R}} (e^{2y} \wedge |y|^2) \varphi(S_t, y) m(t, dy) | \mathcal{F}_{t_0} \right] = \int_{\mathbb{R}} (e^{2y} \wedge |y|^2) \varphi(S_{t_0}, y) m(t_0, dy).$$

**Assumption 3.2** (Integrability condition).

$$\exists T > t_0, \quad \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_0}^T \delta_s^2 ds + \int_{t_0}^T ds \int_{\mathbb{R}} (e^y - 1)^2 m(s, dy) \right) | \mathcal{F}_{t_0} \right] < \infty \quad .$$

We recall that the value  $C_{t_0}(t, K)$  at time  $t_0$  of a call option with expiry  $t > t_0$  and strike  $K > 0$  is given by

$$C_{t_0}(t, K) = e^{-\int_{t_0}^t r(s) ds} \mathbb{E}[\max(S_t - K, 0) | \mathcal{F}_{t_0}]. \quad (25)$$

The discounted asset price

$$\hat{S}_t = e^{-\int_{t_0}^t r(u) du} S_t,$$

is the stochastic exponential of the martingale  $\xi$  defined by

$$\xi_t = \int_0^t \delta_s dW_s + \int_0^t \int (e^y - 1) \tilde{M}(ds dy).$$

Under Assumption 3.2, we have

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \langle \xi, \xi \rangle_T^d + \langle \xi, \xi \rangle_T^c \right) \right] < \infty,$$

where  $\langle \xi, \xi \rangle^c$  and  $\langle \xi, \xi \rangle^d$  denote the continuous and purely discontinuous parts of  $[\xi, \xi]$  and [18, Theorem 9] implies that  $(\hat{S}_t)_{t \in [t_0, T]}$  is a  $\mathbb{P}$ -martingale. In particular the expectation in (25) is finite.

### 3.1 Out-of-the money call options

We first study the asymptotics of out-of-the money call options i.e. the case where  $K > S_{t_0}$ . The main result is as follows:

**Theorem 3.1** (Short-maturity behavior of out-of-the money options). *Under Assumption 3.1 and Assumption 3.2, if  $S_{t_0} < K$  then*

$$\frac{1}{t - t_0} C_{t_0}(t, K) \xrightarrow[t \rightarrow t_0^+]{\quad} \int_0^\infty (S_{t_0} e^y - K)_+ m(t_0, dy). \quad (26)$$

This limit can also be expressed using the exponential double tail  $\psi_{t_0}$  of the compensator, defined as

$$\psi_{t_0}(z) = \int_z^{+\infty} dx e^x \int_x^\infty m(t_0, du) \quad z > 0. \quad (27)$$

Then, as shown in [4, Lemma 1],

$$\int_0^\infty (S_{t_0} e^y - K)_+ m(t_0, dy) = S_{t_0} \psi_{t_0} \left( \ln \left( \frac{K}{S_{t_0}} \right) \right).$$

*Proof.* The idea is to apply Theorem 2.1 to smooth approximations  $f_n$  of the function  $x \rightarrow (x - K)^+$  and conclude using a dominated convergence argument.

First, as argued in the proof of Theorem 2.1, we put  $t_0 = 0$  in the sequel and consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets. Applying the Itô formula to  $X_t \equiv \ln(S_t)$ , we obtain

$$\begin{aligned} X_t &= \ln(S_0) + \int_0^t \frac{1}{S_{s-}} dS_s + \frac{1}{2} \int_0^t \frac{-1}{S_{s-}^2} (S_{s-} \delta_s)^2 ds \\ &\quad + \sum_{s \leq t} \left[ \ln(S_{s-} + \Delta S_s) - \ln(S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right] \\ &= \ln(S_0) + \int_0^t \left( r(s) - \frac{1}{2} \delta_s^2 \right) ds + \int_0^t \delta_s dW_s \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} (e^y - 1) \tilde{M}(ds dy) - \int_0^t \int_{-\infty}^{+\infty} (e^y - 1 - y) M(ds dy) \end{aligned}$$

Note that there exists  $C > 0$  such that

$$|e^y - 1 - y \frac{1}{1 + |y|^2}| \leq C (e^y - 1)^2.$$

Thanks to Jensen's inequality, Assumption 3.2 implies that this quantity is finite, allowing us to write

$$\begin{aligned} &\int_0^t \int_{-\infty}^{+\infty} (e^y - 1) \tilde{M}(ds dy) - \int_0^t \int_{-\infty}^{+\infty} (e^y - 1 - y) M(ds dy) \\ &= \int_0^t \int_{-\infty}^{+\infty} (e^y - 1) \tilde{M}(ds dy) - \int_0^t \int_{-\infty}^{+\infty} \left( e^y - 1 - y \frac{1}{1 + |y|^2} \right) M(ds dy) \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} \left( y - y \frac{1}{1 + |y|^2} \right) M(ds dy) \\ &= \int_0^t \int_{-\infty}^{+\infty} (e^y - 1) \tilde{M}(ds dy) - \int_0^t \int_{-\infty}^{+\infty} \left( e^y - 1 - y \frac{1}{1 + |y|^2} \right) \tilde{M}(ds dy) \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \left( e^y - 1 - y \frac{1}{1 + |y|^2} \right) m(s, y) ds dy + \int_0^t \int_{-\infty}^{+\infty} \left( y - y \frac{1}{1 + |y|^2} \right) M(ds dy) \\ &= \int_0^t \int_{-\infty}^{+\infty} y \frac{1}{1 + |y|^2} \tilde{M}(ds dy) - \int_0^t \int_{-\infty}^{+\infty} \left( e^y - 1 - y \frac{1}{1 + |y|^2} \right) m(s, y) ds dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} \left( y - y \frac{1}{1 + |y|^2} \right) M(ds dy). \end{aligned}$$

We can thus represent  $X_t$  as in (2)):

$$\begin{aligned} X_t = & X_0 + \int_0^t \beta_s dt + \int_0^t \delta_s dW_s \\ & + \int_0^t \int_{-\infty}^{+\infty} y \frac{1}{1+|y|^2} \tilde{M}(ds dy) + \int_0^t \int_{-\infty}^{+\infty} \left( y - y \frac{1}{1+|y|^2} \right) M(ds dy), \end{aligned} \quad (28)$$

with

$$\beta_t = r(t) - \frac{1}{2} \delta_t^2 - \int_{-\infty}^{\infty} \left( e^y - 1 - y \frac{1}{1+|y|^2} \right) m(t, y) dt dy.$$

Hence, if  $\delta$  and  $m(\cdot, dy)$  satisfy Assumption 3.1 then  $\beta$ ,  $\delta$  and  $m(\cdot, dy)$  satisfy Assumption 2.1. Thanks to Jensen's inequality, Assumption 3.2 implies that  $\beta$ ,  $\delta$  and  $m$  satisfy Assumption 2.2. One may apply Theorem 2.1 to  $X_t$  for any function of the form  $f \circ \exp$ ,  $f \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R})$ . Let us introduce a family  $f_n \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R})$  such that

$$\begin{cases} f_n(x) = (x - K)^+ & |x - K| > \frac{1}{n} \\ (x - K)^+ \leq f_n(x) \leq \frac{1}{n} & |x - K| \leq \frac{1}{n}. \end{cases}$$

Then for  $x \neq K$ ,  $f_n(x) \xrightarrow{n \rightarrow \infty} (x - K)^+$ . Define, for  $f \in C_0^\infty(\mathbb{R}^+, \mathbb{R})$ ,

$$\begin{aligned} \mathcal{L}_0 f(x) = & r(0) x f'(x) + \frac{x^2 \delta_0^2}{2} f''(x) \\ & + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1) \cdot f'(x)] m(0, dy). \end{aligned} \quad (29)$$

First, observe that if  $N_1 \geq 1/|S_0 - K|$ ,

$$\forall n \geq N_1, f_n(S_0) = (S_0 - K)^+ = 0, \quad \text{so}$$

$$\frac{1}{t} \mathbb{E} [(S_t - K)^+] \leq \frac{1}{t} \mathbb{E} [f_n(S_t)] = \frac{1}{t} (\mathbb{E} [f_n(S_t)] - f_n(S_0)).$$

Letting  $t \rightarrow 0^+$  yields

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} e^{-\int_0^t r(s) ds} \mathbb{E} [(S_t - K)^+] \leq \mathcal{L}_0 f_n(S_0). \quad (30)$$

Furthermore,

$$\begin{aligned} \mathbb{E} [(S_t - K)^+] & \geq \mathbb{E} [f_n(S_t) 1_{\{|S_t - K| > \frac{1}{n}\}}] \\ & = \mathbb{E} [f_n(S_t)] - \mathbb{E} [f_n(S_t) 1_{\{|S_t - K| \leq \frac{1}{n}\}}] \\ & \geq \mathbb{E} [f_n(S_t)] - f_n(S_0) - \frac{1}{n} \mathbb{E} [1_{\{|S_t - K| \leq \frac{1}{n}\}}]. \end{aligned}$$



But

$$\begin{aligned}\mathbb{E} \left[ 1_{\{|S_t - K| \leq \frac{1}{n}\}} \right] &\leq \mathbb{P} \left( S_t - K \geq -\frac{1}{n} \right) \\ &\leq \mathbb{P} \left( S_t - S_0 \geq K - S_0 - \frac{1}{n} \right).\end{aligned}$$

There exists  $N_2 \geq 0$  such that for all  $n \geq N_2$ ,

$$\begin{aligned}\mathbb{P} \left( S_t - S_0 \geq K - S_0 - \frac{1}{n} \right) &\leq \mathbb{P} \left( S_t - S_0 \geq \frac{K - S_0}{2} \right) \\ &\leq \left( \frac{2}{K - S_0} \right)^2 \mathbb{E} [(S_t - S_0)^2],\end{aligned}$$

by the Bienaymé-Chebyshev inequality. Hence,

$$\frac{1}{t} \mathbb{E} [(S_t - K)^+] \geq \frac{1}{t} (\mathbb{E} [f_n(S_t)] - f_n(S_0)) - \frac{1}{n} \left( \frac{2}{K - S_0} \right)^2 \frac{1}{t} \mathbb{E} [\phi(S_t) - \phi(S_0)],$$

with  $\phi(x) = (x - S_0)^2$ . Applying Theorem 2.1 yields

$$\liminf_{t \rightarrow 0^+} \frac{1}{t} e^{-\int_0^t r(s) ds} \mathbb{E} [(S_t - K)^+] \geq \mathcal{L}_0 f_n(S_0) - \frac{1}{n} \left( \frac{2}{K - S_0} \right)^2 \mathcal{L}_0 \phi(S_0).$$

Letting  $n \rightarrow +\infty$ ,

$$\lim_{t \rightarrow 0^+} \frac{1}{t} e^{-\int_0^t r(s) ds} \mathbb{E} [(S_t - K)^+] = \lim_{n \rightarrow \infty} \mathcal{L}_0 f_n(S_0).$$

Since  $S_0 < K$ ,  $f_n = 0$  in a neighborhood of  $S_0$  for  $n \geq N_1$  so  $f_n(S_0) = f_n''(S_0) = f_n'(S_0) = 0$  and  $\mathcal{L}_0 f_n(S_0)$  reduces to

$$\mathcal{L}_0 f_n(S_0) = \int_{\mathbb{R}} [f_n(S_0 e^y) - f_n(S_0)] m(0, dy).$$

A dominated convergence argument then yields

$$\lim_{n \rightarrow \infty} \mathcal{L}_0 f_n(S_0) = \int_{\mathbb{R}} [(S_0 e^y - K)_+ - (S_0 - K)_+] m(0, dy).$$

Using integration by parts, this last expression may be rewritten [4, Lemma 1] as

$$S_0 \psi_0 \left( \ln \left( \frac{K}{S_0} \right) \right)$$

where  $\psi_0$  is given by (27). This ends the proof.  $\square$

**Remark 3.1.** *Theorem 3.1 also applies to in-the-money options, with a slight modification: for  $K < S_{t_0}$ ,*

$$\frac{1}{t - t_0} (C_{t_0}(t, K) - (S_{t_0} - K)) \xrightarrow[t \rightarrow t_0^+]{} r(t_0) S_{t_0} + S_{t_0} \psi_{t_0} \left( \ln \left( \frac{K}{S_{t_0}} \right) \right), \quad (31)$$

where

$$\psi_{t_0}(z) = \int_{-\infty}^z dx e^x \int_{-\infty}^x m(t_0, du), \quad \text{for } z < 0 \quad (32)$$

denotes the exponential double tail of  $m(0, \cdot)$ .

### 3.2 At-the-money call options

When  $S_{t_0} = K$ , Theorem 3.1 does not apply. Indeed, as already noted in the case of Lévy processes by Tankov [22] and Figueroa-Lopez and Forde [12], the short maturity behavior of at-the-money options depends on whether a continuous martingale component is present and, in absence of such a component, on the degree of activity of small jumps, measured by the Blumenthal-Gettoor index of the Lévy measure which measures its singularity at zero [15]. We will show here that similar results hold in the semimartingale case. We distinguish three cases:

1.  $S$  is a pure jump process of finite variation: in this case at-the-money call options behave linearly in  $t - t_0$  (Proposition 3.1).
2.  $S$  is a pure jump process of infinite variation and its small jumps resemble those of an  $\alpha$ -stable process: in this case at-the-money call options have an asymptotic behavior of order  $|t - t_0|^{1/\alpha}$  when  $t - t_0 \rightarrow 0^+$  (Proposition 3.2).
3.  $S$  has a continuous martingale component which is non-degenerate in the neighborhood of  $t_0$ : in this case at-the-money call options are of order  $\sqrt{t - t_0}$  as  $t \rightarrow t_0^+$ , whether or not jumps are present (Theorem 3.2).

These statements are made precise in the sequel. We observe that, contrarily to the case of out-of-the money options where the presence of jumps dominates the asymptotic behavior, for at-the-money options the presence or absence of a continuous martingale (Brownian) component dominates the asymptotic behavior.

For the finite variation case, we use a slightly modified version of Assumption 3.1:

**Assumption 3.3** (Weak right-continuity of jump compensator). *For all  $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ ,*

$$\lim_{t \rightarrow t_0, t > t_0} \mathbb{E} \left[ \int_{\mathbb{R}} (e^{2y} \wedge |y|) \varphi(S_t, y) m(t, dy) | \mathcal{F}_{t_0} \right] = \int_{\mathbb{R}} (e^{2y} \wedge |y|) \varphi(S_{t_0}, y) m(t_0, dy).$$

**Proposition 3.1** (Asymptotic for ATM call options for pure jump processes of finite variation). *Consider the process*

$$S_t = S_0 + \int_0^t r(s) S_{s-} ds + \int_0^t \int_{-\infty}^{+\infty} S_{s-} (e^y - 1) \tilde{M}(ds dy). \quad (33)$$

Under the Assumptions 3.3 and 3.2 and the condition,

$$\forall t \in [t_0, T], \quad \int_{\mathbb{R}} |y| m(t, dy) < \infty, \quad (34)$$

$$\frac{1}{t - t_0} C_{t_0}(t, S_{t_0}) \xrightarrow[t \rightarrow t_0^+]{\quad} S_{t_0} \int_{\mathbb{R}} (e^y - 1)^+ m(t_0, dy). \quad (35)$$

*Proof.* Replacing  $\mathbb{P}$  by the conditional probability  $\mathbb{P}_{\mathcal{F}_{t_0}}$ , we may set  $t_0 = 0$  in the sequel and consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets. The Tanaka-Meyer formula applied to  $(S_t - S_0)^+$  gives

$$\begin{aligned} (S_t - S_0)^+ &= \int_0^t ds 1_{\{S_{s-} > S_0\}} S_{s-} \left( r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy) \right) \\ &+ \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+. \end{aligned}$$

Hence, applying Fubini's theorem,

$$\begin{aligned} \mathbb{E}[(S_t - S_0)^+] &= \mathbb{E} \left[ \int_0^t ds 1_{\{S_{s-} > S_0\}} S_{s-} \left( r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy) \right) \right] \\ &+ \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} [(S_{s-} e^y - S_0)^+ - (S_{s-} - S_0)^+] m(s, dy) ds \right] \\ &= \int_0^t ds \mathbb{E} \left[ 1_{\{S_s > S_0\}} S_s \left( r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy) \right) \right] \\ &+ \int_0^t ds \mathbb{E} \left[ \int_{\mathbb{R}} [(S_s e^y - S_0)^+ - (S_s - S_0)^+] m(s, dy) \right]. \end{aligned}$$

Since  $\hat{S}$  is a martingale,

$$\mathbb{E}[S_t] = e^{\int_0^t r(s) ds} S_0 < \infty. \quad (36)$$

Hence  $t \rightarrow \mathbb{E}[S_t]$  is right-continuous at 0:

$$\lim_{t \rightarrow 0^+} \mathbb{E}[S_t] = S_0. \quad (37)$$

Furthermore, under the Assumptions 2.1 and 2.2 for  $X_t = \log(S_t)$  (see equation (28)), one may apply Theorem 2.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} [(S_t - S_0)^2] = \mathcal{L}_0 f(X_0),$$

where  $\mathcal{L}_0$  is defined via equation (5). Since  $\mathcal{L}_0 f(X_0) < \infty$ , then in particular,

$$t \mapsto \mathbb{E} [(S_t - S_0)^2]$$

is right-continuous at 0 with right limit 0. Let us show that

$$t \in [0, T[ \mapsto \mathbb{E} \left[ S_t 1_{\{S_t > S_0\}} \left( r(t) - \int_{\mathbb{R}} (e^y - 1) m(t, dy) \right) \right]$$

is right-continuous at 0 with right limit 0. Then applying Lemma 2.1 yields

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} \left[ \int_0^t ds S_s 1_{\{S_s > S_0\}} \left( r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy) \right) \right] = 0.$$

Observing that

$$S_t 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}},$$

we write

$$\begin{aligned} & \left| \mathbb{E} \left[ S_t 1_{\{S_t > S_0\}} \left( r(t) - \int_{\mathbb{R}} (e^y - 1) m(t, dy) \right) \right] \right| \\ &= \left| \mathbb{E} \left[ ((S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}}) \left( r(t) - \int_{\mathbb{R}} (e^y - 1) m(t, dy) \right) \right] \right| \\ &\leq \|r\|_{\infty} \mathbb{E} [|S_t - S_0|] + \|r\|_{\infty} \mathbb{P}(S_t > S_0) \\ &+ \mathbb{E} [(S_t - S_0)^2]^{1/2} \mathbb{E} \left[ \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy) \right]^{1/2} \\ &+ S_0^2 \mathbb{P}[S_t > S_0]^{1/2} \mathbb{E} \left[ \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy) \right]^{1/2}, \end{aligned}$$

using the Lipschitz continuity of  $x \mapsto (x - S_0)_+$  and the Cauchy-Schwarz inequality. Since  $S$  is càdlàg,

$$\lim_{t \downarrow 0} \mathbb{P}(S_t > S_0) = 0,$$

and Assumption 3.3 implies that

$$\lim_{t \downarrow 0} \mathbb{E} \left[ \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy) \right] = \int_{\mathbb{R}} (e^y - 1)^2 m(0, dy) < \infty.$$

Letting  $t \rightarrow 0^+$  in the above inequalities yields the result.

Let us now focus on the jump term and show that

$$t \in [0, T[ \mapsto \mathbb{E} \left[ \int_{\mathbb{R}} [(S_t e^y - S_0)^+ - (S_t - S_0)^+] m(t, dy) \right],$$

is right-continuous at 0 with right-limit

$$S_0 \int_{\mathbb{R}} (e^y - 1)^+ m(0, dy).$$

One shall simply observe that

$$|(xe^y - S_0)^+ - (x - S_0)^+ - (S_0 e^y - S_0)^+| \leq (x + S_0) |e^y - 1|,$$

using the Lipschitz continuity of  $x \mapsto (x - S_0)_+$  and apply Assumption 3.3. This ends the proof.  $\square$

**Proposition 3.2** (Asymptotics of ATM call options for pure-jump martingales of infinite variation). *Consider a semimartingale whose continuous martingale part is zero:*

$$S_t = S_0 + \int_0^t r(s) S_{s-} ds + \int_0^t \int_{-\infty}^{+\infty} S_{s-} (e^y - 1) \tilde{M}(ds dy). \quad (38)$$

Under the Assumptions 3.1 and 3.2, if there exists  $\alpha \in ]1, 2[$  and a family  $m^\alpha(t, dy)$  of positive measures such that

$$\forall t \in [t_0, T], \quad m(\omega, t, dy) = m^\alpha(\omega, t, dy) + 1_{|y| \leq 1} \frac{c(y)}{|y|^{1+\alpha}} dy \text{ a.s.}, \quad (39)$$

where  $c(\cdot) > 0$  is continuous at 0 and

$$\forall t \in [t_0, T] \quad \int_{\mathbb{R}} |y| m^\alpha(t, dy) < \infty, \quad (40)$$

then

$$\frac{1}{(t - t_0)^{1/\alpha}} C_{t_0}(t, S_{t_0}) \xrightarrow[t \rightarrow t_0^+]{} S_{t_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-c(0)|z|^\alpha} - 1}{z^2} dz. \quad (41)$$

*Proof.* Without loss of generality, we set  $t_0 = 0$  in the sequel and consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets. The at-the-money call price can be expressed as

$$C_0(t, S_0) = \mathbb{E}[(S_t - S_0)^+] = S_0 \mathbb{E}\left[\left(\frac{S_t}{S_0} - 1\right)^+\right]. \quad (42)$$

Define, for  $f \in C_b^2([0, \infty[, \mathbb{R})$

$$\mathcal{L}_0 f(x) = r(0) x f'(x) + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1) f'(x)] m(0, dy). \quad (43)$$

We decompose  $\mathcal{L}_0$  as the sum  $\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{J}_0$  where

$$\mathcal{K}_0 f(x) = r(0) x f'(x) + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1) f'(x)] m^\alpha(0, dy),$$

$$\mathcal{J}_0 f(x) = \int_{-1}^1 [f(xe^y) - f(x) - x(e^y - 1) f'(x)] \frac{c(y)}{|y|^{1+\alpha}} dy.$$

The term  $\mathcal{K}_0$  may be interpreted in terms of Theorem 2.1: if  $(Z_t)_{[0,T]}$  is a *finite variation* semimartingale of the form (38) starting from  $Z_0 = S_0$  with jump compensator  $m^\alpha(t, dy)$ , then by Theorem 2.1,

$$\forall f \in C_b^2([0, \infty[, \mathbb{R}), \quad \lim_{t \rightarrow 0^+} \frac{1}{t} e^{-\int_0^t r(s) ds} \mathbb{E}[f(Z_t)] = \mathcal{K}_0 f(S_0). \quad (44)$$

The idea is now to interpret  $\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{J}_0$  in terms of a *multiplicative decomposition*  $S_t = Y_t Z_t$  where  $Y = \mathcal{E}(L)$  is the stochastic exponential of a pure-jump Lévy process with Lévy measure  $c(y)/|y|^{1+\alpha} dy$ , which we can take independent from  $Z$ . Indeed, let  $Y = \mathcal{E}(L)$  where  $L$  is a pure-jump Lévy martingale with Lévy measure  $1_{|y| \leq 1} c(y)/|y|^{1+\alpha} dy$ , independent from  $Z$ , with infinitesimal generator  $\mathcal{J}_0$ . Then  $Y$  is a martingale and  $[Y, Z] = 0$ . Then  $S = YZ$  and  $Y$  is an exponential Lévy martingale, independent from  $Z$ , with  $E[Y_t] = 1$ .

A result of Tankov [22, Proposition 5, Proof 2] for exponential Lévy processes then implies that

$$\frac{1}{t^{1/\alpha}} \mathbb{E}[(Y_t - 1)^+] \xrightarrow{t \rightarrow 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-c(0)|z|^\alpha} - 1}{z^2} dz. \quad (45)$$

We will show that the term (45) is the dominant term which gives the asymptotic behavior of  $C_0(T, S_0)$ .

Indeed, by the Lipschitz continuity of  $x \mapsto (x - S_0)_+$ ,

$$|(S_t - S_0)_+ - S_0(Y_t - 1)_+| \leq Y_t |Z_t - S_0|,$$

so, taking expectations and using that  $Y$  is independent from  $Z$ , we get

$$\underbrace{\mathbb{E}[e^{-\int_0^t r(s) ds} |(S_t - S_0)_+ - S_0(Y_t - 1)_+|]}_{C_0(t, S_0)} \leq \underbrace{\mathbb{E}(Y_t)}_{=1} \mathbb{E}[e^{-\int_0^t r(s) ds} |Z_t - S_0|].$$

To estimate the right hand side of this inequality note that  $|Z_t - S_0| = (Z_t - S_0)_+ + (S_0 - Z_t)_+$ . Since  $Z$  has finite variation, from Proposition 3.1

$$E[e^{-\int_0^t r(s) ds} (Z_t - S_0)_+] \stackrel{t \rightarrow 0^+}{\sim} t S_0 \int_0^\infty dx e^x m([x, +\infty[).$$

Using the martingale property of  $e^{-\int_0^t r(s) ds} Z_t$  yields

$$E[e^{-\int_0^t r(s) ds} (S_0 - Z_t)_+] \stackrel{t \rightarrow 0^+}{\sim} t S_0 \int_0^\infty dx e^x m([x, +\infty[).$$

Hence, dividing by  $t^{1/\alpha}$  and taking  $t \rightarrow 0^+$  we obtain

$$\frac{1}{t^{1/\alpha}} e^{-\int_0^t r(s) ds} \mathbb{E}[|Z_t - S_0|^+] \xrightarrow{t \rightarrow 0^+} 0.$$

Thus, dividing by  $t^{1/\alpha}$  the above inequality and using (45) yields

$$\frac{1}{t^{1/\alpha}} e^{-\int_0^t r(s) ds} \mathbb{E}[(S_t - S_0)_+] \xrightarrow{t \rightarrow 0^+} S_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-c(0)|z|^\alpha} - 1}{z^2} dz.$$

□

We now focus on a third case, when  $S$  is a continuous semimartingale, i.e. an Ito process. From known results in the diffusion case [7], we expect in this case a short-maturity behavior in  $O(\sqrt{t})$ . We propose here a proof of this behavior in a semimartingale setting using the notion of semimartingale local time.

**Proposition 3.3** (Asymptotic for at-the-money options for continuous semimartingales). *Consider the process*

$$S_t = S_0 + \int_0^t r(s)S_s ds + \int_0^t S_s \delta_s dW_s. \quad (46)$$

*Under the Assumptions 3.1 and 3.2 and the following non-degeneracy condition in the neighborhood of  $t_0$ ,*

$$\exists \epsilon > 0, \quad \mathbb{P}(\forall t \in [t_0, T], \quad \delta_t \geq \epsilon) = 1,$$

*we have*

$$\frac{1}{\sqrt{t-t_0}} C_{t_0}(t, S_{t_0}) \xrightarrow[t \rightarrow t_0^+]{\quad} \frac{S_{t_0}}{\sqrt{2\pi}} \delta_{t_0}. \quad (47)$$

*Proof.* Set  $t_0 = 0$  and consider, without loss of generality, the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets. Applying the Tanaka-Meyer formula to  $(S_t - S_0)^+$ , we have

$$(S_t - S_0)^+ = \int_0^t 1_{\{S_s > S_0\}} dS_s + \frac{1}{2} L_t^{S_0}(S).$$

where  $L_t^{S_0}(S)$  corresponds to the semimartingale local time of  $S_t$  at level  $S_0$  under  $\mathbb{P}$ . As noted in Section 3.1, Assumption 3.2 implies that the discounted price  $\hat{S}_t = e^{-\int_0^t r(s) ds} S_t$  is a  $\mathbb{P}$ -martingale. So

$$dS_t = e^{\int_0^t r(s) ds} \left( r(t) S_t dt + d\hat{S}_t \right), \quad \text{and}$$

$$\int_0^t 1_{\{S_s > S_0\}} dS_s = \int_0^t e^{\int_0^s r(u) du} 1_{\{S_s > S_0\}} d\hat{S}_s + \int_0^t e^{\int_0^s r(u) du} r(s) S_s 1_{\{S_s > S_0\}} ds,$$

where the first term is a martingale. Taking expectations, we get:

$$C(t, S_0) = \mathbb{E} \left[ e^{-\int_0^t r(s) ds} \int_0^t e^{\int_0^s r(u) du} r(s) S_s 1_{\{S_s > S_0\}} ds + \frac{1}{2} e^{-\int_0^t r(s) ds} L_t^{S_0}(S) \right].$$

Since  $\hat{S}$  is a martingale,

$$\forall t \in [0, T] \quad \mathbb{E}[S_t] = e^{\int_0^t r(s) ds} S_0 < \infty. \quad (48)$$

Hence  $t \rightarrow \mathbb{E}[S_t]$  is right-continuous at 0:

$$\lim_{t \rightarrow 0^+} \mathbb{E}[S_t] = S_0. \quad (49)$$

Furthermore, under the Assumptions 2.1 and 2.2 for  $X_t = \log(S_t)$  (see equation (28)), one may apply Theorem 2.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} [(S_t - S_0)^2] = \mathcal{L}_0 f(X_0),$$

where  $\mathcal{L}_0$  is defined via equation (5) with  $m \equiv 0$ . Since  $\mathcal{L}_0 f(X_0) < \infty$ , then in particular,

$$t \mapsto \mathbb{E} [(S_t - S_0)^2]$$

is right-continuous at 0 with right limit 0. Let us show that

$$t \in [0, T] \mapsto \mathbb{E} [S_t 1_{\{S_t > S_0\}}]$$

is right-continuous at 0 with right limit 0. Then applying Lemma 2.1 yields

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} \left[ \int_0^t ds S_s 1_{\{S_s > S_0\}} \right] = 0$$

Observing that

$$S_t 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}},$$

we write

$$\begin{aligned} |\mathbb{E} [S_t 1_{\{S_t > S_0\}}]| &= |\mathbb{E} [(S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}}]| \\ &\leq \mathbb{E} [|S_t - S_0|] + \mathbb{P}(S_t > S_0) \end{aligned}$$

using the Lipschitz continuity of  $x \mapsto (x - S_0)_+$ . Since  $S$  is càdlàg,

$$\lim_{t \downarrow 0} \mathbb{P}(S_t > S_0) = 0.$$

Letting  $t \rightarrow 0$  in the above inequalities yields the result. Since

$$\mathbb{E} \left[ \int_0^t e^{\int_0^s r(u) du} r(s) S_s 1_{\{S_s > S_0\}} ds \right] = o(t),$$

a fortiori,

$$\mathbb{E} \left[ \int_0^t e^{\int_0^s r(u) du} r(s) S_s 1_{\{S_s > S_0\}} ds \right] = o(\sqrt{t}).$$

Hence (if the limit exists)

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} C(t, S_0) = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} e^{-\int_0^t r(s) ds} \mathbb{E} \left[ \frac{1}{2} L_t^{S_0}(S) \right] = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \mathbb{E} \left[ \frac{1}{2} L_t^{S_0}(S) \right]. \quad (50)$$



By the Dubins-Schwarz theorem [20, Theorem 1.5], there exists a Brownian motion  $B$  such that

$$\forall t < [U]_\infty, \quad U_t = \int_0^t \delta_s dW_s = B_{[U]_t} = B_{\int_0^t \delta_s^2 ds}.$$

$$\begin{aligned} \text{So } \forall t < [U]_\infty \quad S_t &= S_0 \exp \left( \int_0^t \left( r(s) - \frac{1}{2} \delta_s^2 \right) ds + B_{[U]_t} \right) \\ &= S_0 \exp \left( \int_0^t \left( r(s) - \frac{1}{2} \delta_s^2 \right) ds + B_{\int_0^t \delta_s^2 ds} \right). \end{aligned}$$

The occupation time formula then yields, for  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R})$ ,

$$\begin{aligned} \int_0^\infty \phi(K) L_t^K (S_0 \exp(B_{[U]})) dK &= \int_0^t \phi(S_0 \exp(B_{[U]_u})) S_0^2 \exp(B_{[U]_u})^2 \delta_u^2 du \\ &= \int_{-\infty}^\infty \phi(S_0 \exp(y)) S_0^2 \exp(y)^2 L_t^y(B_{[U]}) dy, \end{aligned}$$

where  $L_t^K(S_0 \exp(B_{[U]}))$  (resp.  $L_t^y(B_{[U]})$ ) denotes the semimartingale local time of the process  $S_0 \exp(B_{[U]})$  at  $K$  and (resp.  $B_{[U]}$  at  $y$ ). A change of variable leads to

$$\begin{aligned} &\int_{-\infty}^\infty \phi(S_0 \exp(y)) S_0 \exp(y) L_t^{S_0 e^y}(S_0 \exp(B_{[U]})) dy \\ &= \int_{-\infty}^\infty \phi(S_0 \exp(y)) S_0^2 \exp(y)^2 L_t^y(B_{[U]}) dy. \end{aligned}$$

Hence

$$L_t^{S_0}(S_0 \exp(B_{[U]})) = S_0 L_t^0(B_{[U]}).$$

We also have

$$L_t^0(B_{[U]}) = L_{\int_0^t \delta_s^2 ds}^0(B),$$

where  $L_{\int_0^t \delta_s^2 ds}^0(B)$  denotes the semimartingale local time of  $B$  at time  $\int_0^t \delta_s^2 ds$  and level 0. Using the scaling property of Brownian motion,

$$\begin{aligned} \mathbb{E} \left[ L_t^{S_0}(S_0 \exp(B_{[U]})) \right] &= S_0 \mathbb{E} \left[ L_{\int_0^t \delta_s^2 ds}^0(B) \right] \\ &= S_0 \mathbb{E} \left[ \sqrt{\int_0^t \delta_s^2 ds} L_1^0(B) \right]. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{t}} \mathbb{E} \left[ L_t^{S_0}(S_0 \exp(B_{[U]})) \right] &= \lim_{t \rightarrow 0^+} \frac{1}{\sqrt{t}} S_0 \mathbb{E} \left[ \sqrt{\int_0^t \delta_s^2 ds} L_1^0(B) \right] \\ &= \lim_{t \rightarrow 0^+} S_0 \mathbb{E} \left[ \sqrt{\frac{1}{t} \int_0^t \delta_s^2 ds} L_1^0(B) \right]. \end{aligned}$$

Let us show that

$$\lim_{t \rightarrow 0^+} S_0 \mathbb{E} \left[ \sqrt{\frac{1}{t} \int_0^t \delta_s^2 ds} L_1^0(B) \right] = S_0 \delta_0 \mathbb{E} [L_1^0(B)]. \quad (51)$$

Using the Cauchy-Schwarz inequality,

$$\left| \mathbb{E} \left[ \left( \sqrt{\frac{1}{t} \int_0^t \delta_s^2 ds} - \delta_0 \right) L_1^0(B) \right] \right| \leq \mathbb{E} [L_1^0(B)^2]^{1/2} \mathbb{E} \left[ \left( \sqrt{\frac{1}{t} \int_0^t \delta_s^2 ds} - \delta_0 \right)^2 \right]^{1/2}.$$

The Lipschitz property of  $x \rightarrow (\sqrt{x} - \delta_0)^2$  on  $[\epsilon, +\infty[$  yields

$$\begin{aligned} \mathbb{E} \left[ \left( \sqrt{\frac{1}{t} \int_0^t \delta_s^2 ds} - \delta_0 \right)^2 \right] &\leq c(\epsilon) \mathbb{E} \left[ \left| \frac{1}{t} \int_0^t (\delta_s^2 - \delta_0^2) ds \right| \right] \\ &\leq \frac{c(\epsilon)}{t} \int_0^t ds \mathbb{E} [|\delta_s^2 - \delta_0^2|]. \end{aligned}$$

where  $c(\epsilon)$  is the Lipschitz constant of  $x \rightarrow (\sqrt{x} - \delta_0)^2$  on  $[\epsilon, +\infty[$ . Assumption 3.1 and Lemma 2.1 then imply (51). By Lévy's theorem for the local time of Brownian motion,  $L_1^0(B)$  has the same law as  $|B_1|$ , leading to

$$\mathbb{E} [L_1^0(B)] = \sqrt{\frac{2}{\pi}}.$$

Clearly, since  $L_t^K(S) = L_t^K(S_0 \exp(B_{[t]}))$ ,

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \mathbb{E} \left[ \frac{1}{2} L_t^{S_0}(S) \right] = \frac{S_0}{\sqrt{2\pi}} \delta_0. \quad (52)$$

This ends the proof.  $\square$

We can now treat the case of a general Itô semimartingale with both a continuous martingale component and a jump component.

**Theorem 3.2** (Short-maturity asymptotics for at-the-money call options). *Consider the price process  $S$  whose dynamics is given by*

$$S_t = S_0 + \int_0^t r(s) S_{s-} ds + \int_0^t S_{s-} \delta_s dW_s + \int_0^t \int_{-\infty}^{+\infty} S_{s-} (e^y - 1) \tilde{M}(ds dy).$$

*Under the Assumptions 3.1 and 3.2 and the following non-degeneracy condition in the neighborhood of  $t_0$*

$$\exists \epsilon > 0, \quad \mathbb{P}(\forall t \in [t_0, T], \quad \delta_t \geq \epsilon) = 1,$$

*we have*

$$\frac{1}{\sqrt{t - t_0}} C_{t_0}(t, S_{t_0}) \xrightarrow[t \rightarrow t_0^+]{\quad} \frac{S_{t_0}}{\sqrt{2\pi}} \delta_{t_0}. \quad (53)$$

*Proof.* Applying the Tanaka-Meyer formula to  $(S_t - S_0)^+$ , we have

$$\begin{aligned} (S_t - S_0)^+ &= \int_0^t 1_{\{S_{s-} > S_0\}} dS_s + \frac{1}{2} L_t^{S_0} \\ &\quad + \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s. \end{aligned} \quad (54)$$

As noted above, Assumption 3.2 implies that the discounted price  $\hat{S}_t = e^{-\int_0^t r(s) ds} S_t$  is a martingale under  $\mathbb{P}$ . So  $(S_t)$  can be expressed as  $dS_t = e^{\int_0^t r(s) ds} \left( r(t) S_{t-} dt + d\hat{S}_t \right)$  and

$$\int_0^t 1_{\{S_{s-} > S_0\}} dS_s = \int_0^t e^{\int_0^s r(u) du} 1_{\{S_{s-} > S_0\}} d\hat{S}_s + \int_0^t e^{\int_0^s r(u) du} r(s) S_{s-} 1_{\{S_{s-} > S_0\}} ds,$$

where the first term is a martingale. Taking expectations, we get

$$\begin{aligned} e^{\int_0^t r(s) ds} C(t, S_0) &= \mathbb{E} \left[ \int_0^t e^{\int_0^s r(u) du} r(s) S_s 1_{\{S_{s-} > S_0\}} ds + \frac{1}{2} L_t^{S_0} \right] \\ &\quad + \mathbb{E} \left[ \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s \right]. \end{aligned}$$

Since  $\hat{S}$  is a martingale,

$$\forall t \in [0, T] \quad \mathbb{E}[S_t] = e^{\int_0^t r(s) ds} S_0 < \infty. \quad (55)$$

Hence  $t \rightarrow \mathbb{E}[S_t]$  is right-continuous at 0:

$$\lim_{t \rightarrow 0^+} \mathbb{E}[S_t] = S_0. \quad (56)$$

Furthermore, under the Assumptions 2.1 and 2.2 for  $X_t = \log(S_t)$  (see equation (28)), one may apply Theorem 2.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E}[(S_t - S_0)^2] = \mathcal{L}_0 f(X_0),$$

where  $\mathcal{L}_0$  is defined via equation (5). Since  $\mathcal{L}_0 f(X_0) < \infty$ , then in particular,

$$t \mapsto \mathbb{E}[(S_t - S_0)^2]$$

is right-continuous at 0 with right limit 0. Let us show that

$$t \in [0, T] \mapsto \mathbb{E}[S_t 1_{\{S_t > S_0\}}]$$

is right-continuous at 0 with right limit 0. Then applying Lemma 2.1 yields

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E} \left[ \int_0^t ds S_s 1_{\{S_s > S_0\}} \right] = 0$$

Observing that

$$S_t 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}},$$

we write

$$|\mathbb{E} [S_t 1_{\{S_t > S_0\}}]| \leq \mathbb{E} [|S_t - S_0|] + S_0 \mathbb{P}(S_t > S_0)$$

Letting  $t \rightarrow 0^+$  in the above inequalities yields

$$\mathbb{E} \left[ \int_0^t r(s) S_s 1_{\{S_{s-} > S_0\}} ds \right] = o(t) = o(\sqrt{t}).$$

Let us now focus on the jump part,

$$\begin{aligned} & \mathbb{E} \left[ \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s \right] \\ &= \mathbb{E} \left[ \int_0^t ds \int m(s, dx) (S_{s-} e^x - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} S_{s-} (e^x - 1) \right] \end{aligned} \quad (57)$$

Observing that

$$|(ze^x - S_0)^+ - (z - S_0)^+ - 1_{\{z > S_0\}} z(e^x - 1)| \leq C (S_0 e^x - z)^2,$$

then, together with Assumption 3.1 and Lemma 2.1 implies,

$$\mathbb{E} \left[ \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s \right] = O(t) = o(\sqrt{t}).$$

Since  $\delta_0 \geq \epsilon$ , equation (52) yields the result.  $\square$

**Remark 3.2.** As noted by Berestycki et al [6, 8] in the diffusion case, the regularity of  $f$  at  $S_{t_0}$  plays a crucial role in the asymptotics of  $\mathbb{E} [f(S_t)]$ . Theorem 2.1 shows that  $\mathbb{E} [f(S_t)] \sim ct$  for smooth functions  $f$ , even if  $f(S_{t_0}) = 0$ , while for call option prices we have  $\sim \sqrt{t}$  asymptotics at-the-money where the function  $x \rightarrow (x - S_0)_+$  is not smooth.

**Remark 3.3.** In the particular case of a Lévy process, Proposition 3.1, Proposition 3.2 and Theorem 3.2 imply [22, Proposition 5, Proof 2].

## References

- [1] E. ALÒS, J. A. LEÓN, AND J. VIVES, *On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility*, Finance Stoch., 11 (2007), pp. 571–589.
- [2] O. BARNDORFF-NIELSEN AND F. HUBALEK, *Probability measures, Lévy measures and analyticity in time*, Bernoulli, 14 (2008), pp. 764–790.
- [3] E. BENHAMOU, E. GOBET, AND M. MIRI, *Smart expansion and fast calibration for jump diffusions*, Finance Stoch., 13 (2009), pp. 563–589.
- [4] A. BENTATA AND R. CONT, *Forward equations for option prices in semimartingale models*, (2009), forthcoming in: Finance and Stochastics.
- [5] A. BENTATA AND R. CONT, *Mimicking the marginal distributions of a semimartingale*, Working paper, arXiv:0910.3992v2 [math.PR], 2009.
- [6] H. BERESTYCKI, J. BUSCA, AND I. FLORENT, *Asymptotics and calibration of local volatility models*, Quant. Finance, 2 (2002), pp. 61–69. Special issue on volatility modelling.
- [7] H. BERESTYCKI, J. BUSCA, AND I. FLORENT, *Asymptotics and calibration of local volatility models*, Quantitative Finance, 2 (2002), pp. 61–69.
- [8] H. BERESTYCKI, J. BUSCA, AND I. FLORENT, *Computing the implied volatility in stochastic volatility models*, Comm. Pure Appl. Math., 57 (2004), pp. 1352–1373.
- [9] V. DURRLEMAN, *Convergence of at-the-money implied volatilities to the spot volatility*, Journal of Applied Probability, 45 (2008), pp. 542–550.
- [10] ———, *From implied to spot volatilities*, Finance and Stochastics, 14 (2010), pp. 157–177.
- [11] J. FENG, M. FORDE, AND J.-P. FOUQUE, *Short-maturity asymptotics for a fast mean-reverting Heston stochastic volatility model*, SIAM Journal of Financial Mathematics, 1 (2010), pp. 126–141.
- [12] J. E. FIGUEROA-LÓPEZ AND M. FORDE, *The small-maturity smile for exponential Lévy models*, SIAM Journal of Financial Mathematics, 3 (2012), pp. 33–65.
- [13] J. E. FIGUEROA-LÓPEZ AND C. HOUDRÉ, *Small-time expansions for the transition distributions of Lévy processes*, Stochastic Process. Appl., 119 (2009), pp. 3862–3889.
- [14] J. GATHERAL, E. P. HSU, P. LAURENCE, C. OUYANG, AND T.-H. WANG, *Asymptotics of implied volatility in local volatility models*, Mathematical Finance, (2011).

- [15] J. JACOD, *Asymptotic properties of power variations of Lévy processes*, ESAIM Probab. Stat., 11 (2007), pp. 173–196.
- [16] R. LÉANDRE, *Densité en temps petit dun processus de sauts*, in Séminaire de Probabilités, XXI (Univ. Strasbourg, vol. 1247 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 81–99.
- [17] J. MUHLE-KARBE AND M. NUTZ, *Small-Time Asymptotics of Option Prices and First Absolute Moments*, J. Appl. Probab., 48 (2011), pp. 1003–1020.
- [18] P. PROTTER AND K. SHIMBO, *No arbitrage and general semimartingales*. Ethier, Stewart N. (ed.) et al., Markov processes and related topics: A Festschrift for Thomas G. Kurtz. Beachwood, OH. Institute of Mathematical Statistics Collections 4, 267-283, 2008.
- [19] P. E. PROTTER, *Stochastic integration and differential equations*, Springer-Verlag, Berlin, 2005. Second edition.
- [20] D. REVUZ AND M. YOR, *Continuous martingales and Brownian motion*, vol. 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, third ed., 1999.
- [21] L. RÜSCHENDORF AND J. WOERNER, *Expansion of transition distributions of lévy processes in small time*, Bernoulli, 8 (2002), pp. 81–96.
- [22] P. TANKOV, *Pricing and Hedging in Exponential Lévy Models: Review of Recent Results*, Paris-Princeton Lectures on Mathematical Finance 2010, 2003/2011 (2011), pp. 319–359.